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Martingale-Type Convergence of Modular Automorphism Groups on von Neumann Algebras

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The strong convergence of modular automorphism groups and Connes cocycle derivatives is discussed under an increasing or decreasing net of von Neumann subalgebras and faithful semifinite normal weights on a von Neumann algebra.

INTRODUCTION

Since Tomita–Takesaki theory [17] and Connes theory [7], modular automorphism groups and Connes cocycle derivatives have played a central role in analyzing von Neumann algebras. The strong convergence of associated modular automorphism groups was obtained under the norm convergence of faithful normal positive linear functionals [2] and the monotone increasing convergence of faithful semifinite normal weights [9]. On the other hand, Araki [3, 4] established the convergence of modular operators given an increasing net of von Neumann subalgebras and a faithful normal positive linear functional and applied it to the convergence of relative entropies. On the same lines, we discuss in this paper the martingale type convergence of modular automorphism groups and Connes cocycle derivatives under an increasing or decreasing net of von Neumann subalgebras and faithful semifinite normal weights.

Given a faithful semifinite normal weight φ on a von Neumann algebra M, let Δ_φ and σ_t^φ be the associated modular operator and modular automorphism group. In Section 1 we consider the strong convergence of modular automorphism groups $\sigma_t^{\varphi_\alpha}$ associated with $\varphi_\alpha = \varphi \upharpoonright M_\alpha$ for increasing von Neumann subalgebras $M_\alpha \nearrow M$. It is proved that $\sigma_t^{\varphi_\alpha}(x) \to \sigma_t^\varphi(x)$ strongly for every $x \in \bigcup_\alpha M_\alpha$ if and only if the union of the left Hilbert algebras associated with φ_α is a core of $\Delta_\varphi^{1/2}$. In Section 2 we consider the decreasing case of unital von Neumann subalgebras $M_\alpha \searrow M_\infty$. Under suitable assumptions we show that $\sigma_t^{\varphi_\alpha}(x) \to \sigma_t^{\varphi_\infty}(x)$ strongly for every

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$x \in M_\infty$ where $\varphi_\infty = \varphi \upharpoonright M_\infty$. In Section 3 the strong convergence of Connes cocycle derivatives is obtained in the similar situation of increasing or decreasing von Neumann subalgebras. Finally in Section 4 we give some discussions in connection with the martingale convergence in von Neumann algebras. In this direction we have given further discussions in [12].

1. Increasing Case

Let M be a von Neumann algebra and φ be a faithful semifinite normal weight on M. We use the usual notations $\mathfrak{n}_\varphi = \{x \in M : \varphi(x^*x) < +\infty\}$ and $\mathfrak{m}_\varphi = \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$. Let $(\mathscr{H}_\varphi, \pi_\varphi)$ be the GNS representation of M induced by φ where the canonical injection of $\mathfrak{n}_\varphi$ into $\mathscr{H}_\varphi$ is denoted by $x \mapsto x_\varphi$. Then $\mathfrak{A}_\varphi = \{x_\varphi : x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*\}$ is an achieved left Hilbert algebra and $\pi_\varphi(M)$ is its left von Neumann algebra. Let Δ_φ, J_φ, and σ_t^φ be the modular operator, the modular conjugation, and the modular automorphism group associated with φ.

In this section we discuss the convergence of modular automorphism groups for the increasing case. We fix an increasing net $\{M_\alpha\}$ of von Neumann subalgebras of M with $M = \bigvee_\alpha M_\alpha$ (we write $M_\alpha \nearrow M$). Assume that $\varphi_\alpha = \varphi \upharpoonright M_\alpha$ is semifinite for each α. For each α, we take $\mathfrak{n}_\alpha = \mathfrak{n}_\varphi \cap M_\alpha$, $\mathfrak{m}_\alpha = \mathfrak{n}_\alpha^* \mathfrak{n}_\alpha$, the GNS representation $(\mathscr{H}_\alpha, \pi_\alpha)$ of M_α, the left Hilbert algebra $\mathfrak{A}_\alpha = \{x_\varphi : x \in \mathfrak{n}_\alpha \cap \mathfrak{n}_\alpha^*\}$, the modular operator Δ_α, and the modular automorphism group $\sigma_t^{\varphi_\alpha}$ associated with φ_α. Then $\{\mathscr{H}_\alpha\}$ is naturally an increasing net of subspaces of $\mathscr{H}_\varphi$ where the canonical injection of $\mathfrak{n}_\alpha$ into $\mathscr{H}_\alpha$ is taken as the restriction of $x \mapsto x_\varphi$ to $\mathfrak{n}_\alpha$. Let P_α be the orthogonal projection of $\mathscr{H}_\varphi$ onto $\mathscr{H}_\alpha$, then $P_\alpha \in \pi_\varphi(M_\alpha)'$ and $\pi_\alpha(x) = \pi_\varphi(x) P_\alpha$ for all $x \in M_\alpha$.

Under the above assumptions and notations, we have

THEOREM 1.1. *The following conditions are equivalent*:

(i) $\bigcup_\alpha \mathfrak{A}_\alpha (\subset \mathfrak{A}_\varphi)$ *is a core of* $\Delta_\varphi^{1/2}$;

(ii) $\sigma_t^{\varphi_\alpha}(x)$ *converges strongly to* $\sigma_t^\varphi(x)$ *for every* $x \in \bigcup_\alpha M_\alpha$.

In this case, the convergence in (ii) *is uniform for* t *in any finite interval. Particularly if* φ *is bounded* (*i.e.*, $\varphi(1) < +\infty$), *then conditions* (i) *and* (ii) *hold.*

For the proof of (i) $\Rightarrow$ (ii), we present some lemmas following Araki [3] who dealt with the case $\varphi(1) < +\infty$.

Let $f \in M_*^+$ with $f \leqslant \varphi$ be given. It is known (cf. [6] or [16, Corollary 2.4]) that there exist a unique $T \in \pi_\varphi(M)'$ with $0 \leqslant T \leqslant 1$ and a unique $\eta \in \mathscr{H}_\varphi$ such that

$$f(x) = \langle \pi_\varphi(x)\,\eta, \eta\rangle, \qquad x \in \mathfrak{m}_\varphi,$$
$$Tx_\varphi = \pi_\varphi(x)\,\eta, \qquad x \in \mathfrak{n}_\varphi.$$

By [21, Theorem 3.2], there exists a unique $h \in \mathfrak{m}_\varphi$ with $0 \leqslant h \leqslant 1$ such that

$$f(x) = \tfrac{1}{2}\varphi(hx + xh), \qquad x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*.$$

For each α we further take a unique $h_\alpha \in \mathfrak{m}_\alpha$ with $0 \leqslant h_\alpha \leqslant 1$ such that

$$f(x) = \tfrac{1}{2}\varphi(h_\alpha x + xh_\alpha), \qquad x \in \mathfrak{n}_\alpha \cap \mathfrak{n}_\alpha^*.$$

Then

LEMMA 1.2. (1) *$h_\varphi \in D(\Delta_\varphi)$ and $\Delta_\varphi h_\varphi = 2T\eta - h_\varphi$.*

(2) *For each α, $(h_\alpha)_\varphi \in D(\Delta_\alpha)$ and $\Delta_\alpha(h_\alpha)_\varphi = 2P_\alpha T\eta - (h_\alpha)_\varphi$.*

Proof. (1) For every $x_\varphi \in \mathfrak{A}_\varphi$, we have

$$\begin{aligned} 2\langle x_\varphi, T\eta\rangle &= 2\langle \pi_\varphi(x)\,\eta, \eta\rangle = 2f(x) \\ &= \varphi(hx + xh) = \langle x_\varphi, h_\varphi\rangle + \langle h_\varphi, (x^*)_\varphi\rangle \\ &= \langle x_\varphi, h_\varphi\rangle + \langle \Delta_\varphi^{1/2} x_\varphi, \Delta_\varphi^{1/2} h_\varphi\rangle, \end{aligned}$$

and hence

$$\langle \Delta_\varphi^{1/2} x_\varphi, \Delta_\varphi^{1/2} h_\varphi\rangle = \langle x_\varphi, 2T\eta - h_\varphi\rangle.$$

Since $\mathfrak{A}_\varphi$ is a core of $\Delta_\varphi^{1/2}$, we get $h_\varphi \in D(\Delta_\varphi)$ and $\Delta_\varphi h_\varphi = 2T\eta - h_\varphi$.

(2) As T and η for f, we take $T_\alpha \in \pi_\alpha(M_\alpha)' = P_\alpha \pi_\varphi(M_\alpha)' P_\alpha$ and $\eta_\alpha \in \mathscr{H}_\alpha = P_\alpha \mathscr{H}_\varphi$ for $f \restriction M_\alpha$. We then get $(h_\alpha)_\varphi \in D(\Delta_\alpha)$ and $\Delta_\alpha(h_\alpha)_\varphi = 2T_\alpha \eta_\alpha - (h_\alpha)_\varphi$. It now suffices to show that $T_\alpha \eta_\alpha = P_\alpha T\eta$. For every $x \in \mathfrak{n}_\alpha$, we have

$$\begin{aligned} \langle x_\varphi, T_\alpha \eta_\alpha\rangle &= \langle \pi_\alpha(x)\,\eta_\alpha, \eta_\alpha\rangle = f(x) \\ &= \langle \pi_\varphi(x)\,\eta, \eta\rangle = \langle x_\varphi, T\eta\rangle = \langle x_\varphi, P_\alpha T\eta\rangle, \end{aligned}$$

and hence $T_\alpha \eta_\alpha = P_\alpha T\eta$. Q.E.D.

LEMMA 1.3. *Suppose condition* (i) *in Theorem* 1.1. *Then $\|(h_\alpha)_\varphi - h_\varphi\| \to 0$ and $\|\Delta_\alpha(h_\alpha)_\varphi - \Delta_\varphi h_\varphi\| \to 0$.*

Proof. Condition (i) obviously yields that $P_\alpha \nearrow 1$, so that $\|P_\alpha T\eta - T\eta\| \to 0$. Hence it suffices by Lemma 1.2 to prove the first assertion. Noting that

$$\|(h_\alpha)_\varphi\|^2 = \varphi(h_\alpha^2) = f(h_\alpha) \leqslant \|f\|,$$

let $\xi \in \mathscr{H}_\varphi$ be any weak accumulation point of $\{(h_\alpha)_\varphi\}$. By weak compactness of the unit ball of M, there exists a subnet $\{h_{\alpha'}\}$ of $\{h_\alpha\}$ such that $(h_{\alpha'})_\varphi \to \xi$ weakly in $\mathscr{H}_\varphi$ and $h_{\alpha'} \to k$ weakly for some $k \in M_+$. Then it follows (cf. [16, p. 28]) that $k \in \mathfrak{n}_\varphi$ and $\xi = k_\varphi$. Since

$$2f(x) = \langle x_\varphi, (h_{\alpha'})_\varphi \rangle + \langle (h_{\alpha'})_\varphi, (x^*)_\varphi \rangle, \qquad x_\varphi \in \mathfrak{A}_{\alpha'},$$

we have

$$2f(x) = \langle x_\varphi, k_\varphi \rangle + \langle k_\varphi, (x^*)_\varphi \rangle, \qquad x_\varphi \in \bigcup_\alpha \mathfrak{A}_\alpha.$$

By condition (i), for every $x_\varphi \in \mathfrak{A}_\varphi$ there is a sequence $\{(x_n)_\varphi\}$ in $\bigcup_\alpha \mathfrak{A}_\alpha$ such that $\|(x_n)_\varphi - x_\varphi\| \to 0$ and $\|(x_n^*)_\varphi - (x^*)_\varphi\| \to 0$. Since $f \leqslant \varphi$,

$$\begin{aligned}|f(x_n - x)| &\leqslant f(1)^{1/2} f((x_n - x)^*(x_n - x))^{1/2} \\ &\leqslant f(1)^{1/2} \|(x_n)_\varphi - x_\varphi\| \to 0.\end{aligned}$$

We thus have

$$2f(x) = \langle x_\varphi, k_\varphi \rangle + \langle k_\varphi, (x^*)_\varphi \rangle, \qquad x_\varphi \in \mathfrak{A}_\varphi,$$

so that

$$-\langle x_\varphi, (h-k)_\varphi \rangle = \langle (h-k)_\varphi, (x^*)_\varphi \rangle = \langle \Delta_\varphi^{1/2} x_\varphi, \Delta_\varphi^{1/2} (h-k)_\varphi \rangle, \qquad x_\varphi \in \mathfrak{A}_\varphi.$$

This implies that $(h-k)_\varphi \in D(\Delta_\varphi)$ and $\Delta_\varphi (h-k)_\varphi = -(h-k)_\varphi$. Hence $(h-k)_\varphi = 0$ and so $\xi = k_\varphi = h_\varphi$. Thus $(h_\alpha)_\varphi \to h_\varphi$ weakly in $\mathscr{H}_\varphi$, and moreover $h_\alpha \to h$ weakly in M because h is a single weak accumulation point of $\{h_\alpha\}$. Hence

$$\|(h_\alpha)_\varphi\|^2 = f(h_\alpha) \to f(h) = \|h_\varphi\|^2,$$

so that $\|(h_\alpha)_\varphi - h_\varphi\| \to 0$. Q.E.D.

LEMMA 1.4. *The set $\{(1 + \Delta_\varphi) h_\varphi\}$ is total in $\mathscr{H}_\varphi$ where $h \in \mathfrak{m}_\varphi$ with $0 \leqslant h \leqslant 1$ is taken as above for any $f \in M_*^+$ with $f \leqslant \varphi$.*

Proof. For each $a \in \mathfrak{m}_\varphi$ with $0 \leqslant a \leqslant 1$, define $T \in \pi_\varphi(M)'$ with $0 \leqslant T \leqslant 1$ and $\eta \in \mathscr{H}_\varphi$ by

$$T = J_\varphi \pi_\varphi(a^{1/2}) J_\varphi, \qquad \eta = J_\varphi (a^{1/2})_\varphi.$$

Then

$$T x_\varphi = J_\varphi \pi_\varphi(a^{1/2}) J_\varphi x_\varphi = \pi_\varphi(x) J_\varphi (a^{1/2})_\varphi = \pi_\varphi(x) \eta \qquad x \in \mathfrak{n}_\varphi.$$

Now define $f \in M_*^+$ by

$$f(x) = \langle \pi_\varphi(x)\,\eta, \eta\rangle, \qquad x \in M.$$

If $x \in \mathfrak{m}_\varphi$ with $x \geqslant 0$, then

$$\begin{aligned} f(x) &= \langle Tx_\varphi, \eta\rangle = \langle T\pi_\varphi(x^{1/2})(x^{1/2})_\varphi, \eta\rangle \\ &= \langle T(x^{1/2})_\varphi, \pi_\varphi(x^{1/2})\,\eta\rangle = \| T(x^{1/2})_\varphi\|^2 \\ &\leqslant \|(x^{1/2})_\varphi\|^2 = \varphi(x), \end{aligned}$$

and hence $f \leqslant \varphi$. For this f we take an $h \in \mathfrak{m}_\varphi$ with $0 \leqslant h \leqslant 1$. By Lemma 1.2,

$$(1 + \Delta_\varphi)\,h_\varphi = 2T\eta = 2J_\varphi \pi_\varphi(a^{1/2})(a^{1/2})_\varphi = 2J_\varphi a_\varphi.$$

Since $\{a_\varphi : a \in \mathfrak{m}_\varphi\}$ is dense in $\mathscr{H}_\varphi$, the set

$$\{2J_\varphi a_\varphi : a \in \mathfrak{m}_\varphi, 0 \leqslant a \leqslant 1\}$$

is total in $\mathscr{H}_\varphi$. Q.E.D.

For the proof of (ii) $\Rightarrow$ (i) in Theorem 1.1, we present the following

LEMMA 1.5. *There exists a faithful semifinite normal weight φ_0 on M such that*

(1) $\varphi \leqslant \varphi_0$,
(2) $\varphi_0 \restriction M_\alpha = \varphi_\alpha$ *for each* α,
(3) $\bigcup_\alpha \mathfrak{A}_\alpha$ *is a core of* $\Delta_{\varphi_0}^{1/2}$.

Proof. Let $\mathfrak{A}_0 = \bigcup_\alpha \mathfrak{A}_\alpha$ which is a left Hilbert subalgebra of $\mathfrak{A}_\varphi$. Let $\mathscr{H}_0$ be the completion of $\mathfrak{A}_0$ in $\mathscr{H}_\varphi$ and P_0 the orthogonal projection of $\mathscr{H}_\varphi$ onto $\mathscr{H}_0$. Since $P_\alpha \in \pi_\varphi(M_\alpha)'$ and $P_\alpha \nearrow P_0$, we get $P_0 \in \pi_\varphi(M)'$. Let π_0, π_0', $\mathscr{L}(\mathfrak{A}_0)$, and $\mathfrak{A}_0''$ be the representations by left and right multiplications, the left von Neumann algebra, and the achieved left Hilbert algebra, respectively, associated with $\mathfrak{A}_0$. For each $\eta = x_\varphi \in \mathfrak{A}_0$, we have $\pi_0(\eta) = \pi_\varphi(x)\,P_0$. Hence $\mathscr{L}(\mathfrak{A}_0) = \pi_\varphi(M)\,P_0$. If $\pi_\varphi(x)\,P_0 = 0$ for $x \in M$, then $xy = 0$ for all $y \in \bigcup_\alpha(\mathfrak{n}_\alpha \cap \mathfrak{n}_\alpha^*)$ so that $x = 0$. Hence M is isomorphic to $\mathscr{L}(\mathfrak{A}_0)$ by $x \mapsto \pi_\varphi(x)\,P_0$. Identifying M with $\mathscr{L}(\mathfrak{A}_0)$, we define the associated faithful semifinite normal weight φ_0 on M, i.e., for $a \in M_+$,

$$\begin{aligned} \varphi_0(a) &= \|\xi\|^2 \quad \text{if} \quad \pi_\varphi(a^{1/2})P_0 = \pi_0(\xi) \quad \text{with} \quad \xi \in \mathfrak{A}_0'', \\ &= +\infty \quad \text{otherwise.} \end{aligned}$$

Now let $a \in M_+$ and $\xi \in \mathfrak{A}_0''$ be such that $\pi_\varphi(a^{1/2})P_0 = \pi_0(\xi)$. By [11, Theorem 4], there exists a sequence $\{\xi_n\} = \{(x_n)_\varphi\}$ in $\mathfrak{A}_0$ such that

$$\|(x_n)_\varphi - \xi\| = \|\xi_n - \xi\| \to 0,$$
$$\|(x_n^*)_\varphi - \xi^\#\| = \|\xi_n^\# - \xi^\#\| \to 0,$$
$$\|x_n\| = \|\pi_0(\xi_n)\| \leqslant \|\pi_0(\xi)\|.$$

It then follows that $\xi \in \mathfrak{A}_\varphi$ and hence $\xi = x_\varphi$ with $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$. For every $y \in \bigcup_\alpha (\mathfrak{n}_\alpha \cap \mathfrak{n}_\alpha^*)$, we have $a^{1/2}y = xy$ since

$$(a^{1/2}y)_\varphi = \pi_0(\xi)\, y_\varphi = \lim_n \pi_0'(y_\varphi)\, \xi_n = \lim_n \pi_\varphi(x_n)\, y_\varphi = (xy)_\varphi.$$

Hence $a^{1/2} = x$ and $\varphi_0(a) = \|x_\varphi\|^2 = \varphi(a)$. Moreover, for each $a \in (M_\alpha)_+$ with $\varphi(a) < +\infty$, we get $(a^{1/2})_\varphi \in \mathfrak{A}_0$ and $\varphi_0(a) = \|(a^{1/2})_\varphi\|^2 = \varphi(a)$. We thus deduce properties (1) and (2). Property (3) is obvious from the definition of φ_0. Q.E.D.

Proof of Theorem 1.1. (i) $\Rightarrow$ (ii). For each α, let $\tilde{\Delta}_\alpha = \Delta_\alpha P_\alpha + (1 - P_\alpha)$. For any h_φ as in Lemma 1.2, we have

$$\begin{aligned} &\|\{(1+\tilde{\Delta}_\alpha)^{-1} - (1+\Delta_\varphi)^{-1}\}(1+\Delta_\varphi)\, h_\varphi\| \\ &\quad = \|(1+\tilde{\Delta}_\alpha)^{-1}\{(1+\Delta_\varphi)\, h_\varphi - (1+\Delta_\alpha)(h_\alpha)_\varphi\} + (h_\alpha)_\varphi - h_\varphi\| \\ &\quad \leqslant 2\,\|(h_\alpha)_\varphi - h_\varphi\| + \|\Delta_\alpha (h_\alpha)_\varphi - \Delta_\varphi h_\varphi\| \to 0 \end{aligned}$$

by Lemma 1.3. It follows from Lemma 1.4 that $(1+\tilde{\Delta}_\alpha)^{-1} \to (1+\Delta_\varphi)^{-1}$ strongly. This implies (cf. [15, Problems 21, 27]) that $\tilde{\Delta}_\alpha^{it} \to \Delta_\varphi^{it}$ strongly where the convergence is uniform for t in any finite interval. For every $x \in M_\alpha$, we have

$$\pi_\varphi(\sigma_t^{\varphi_\alpha}(x))\, P_\alpha = \Delta_\alpha^{it} \pi_\varphi(x)\, \Delta_\alpha^{-it} P_\alpha = \tilde{\Delta}_\alpha^{it} \pi_\varphi(x)\, \tilde{\Delta}_\alpha^{-it} P_\alpha.$$

For every $x \in M_{\alpha_0}$, $\xi \in \mathscr{H}_\varphi$, and $\alpha \geqslant \alpha_0$, we have

$$\begin{aligned} &\|\pi_\varphi(\sigma_t^{\varphi_\alpha}(x))\, \xi - \pi_\varphi(\sigma_t^\varphi(x))\, \xi\| \\ &\quad \leqslant \|\tilde{\Delta}_\alpha^{it} \pi_\varphi(x)\, \tilde{\Delta}_\alpha^{-it} P_\alpha \xi - \Delta_\varphi^{it} \pi_\varphi(x)\, \Delta_\varphi^{-it} \xi\| + \|x\|\, \|P_\alpha \xi - \xi\| \end{aligned}$$

which tends to 0 uniformly for t in any finite interval.

(ii) $\Rightarrow$ (i). Let φ_0 be as in Lemma 1.5. By properties (2) and (3), it follows from the part (i) $\Rightarrow$ (ii) that $\sigma_t^{\varphi_\alpha}(x) \to \sigma_t^{\varphi_0}(x)$ strongly for every $x \in \bigcup_\alpha M_\alpha$. Together with assumption (ii), we get $\sigma_t^{\varphi_0}(x) = \sigma_t^\varphi(x)$ for all $x \in M$. By [14, Theorem 5.4], there exists a positive selfadjoint operator z affiliated with the center of M such that $\varphi = \varphi_0(z\, \cdot)$. By property (1), we get

$0 \leqslant z \leqslant 1$ (cf. [16, Proposition 4.5]). Hence (2) implies that $(1-z)\,a = 0$ for every positive $a \in \bigcup_\alpha \mathfrak{m}_\alpha$. Therefore $z = 1$ and $\varphi = \varphi_0$, so that (ii) $\Rightarrow$ (i).

Finally let φ be bounded. For each $x \in M$, taking a net $\{x_j\}$ in $\bigcup_\alpha M_\alpha$ such that $x_j \to x$ strongly*, we have $\|(x_j)_\varphi - x_\varphi\| \to 0$ and $\|(x_j^*)_\varphi - (x^*)_\varphi\| \to 0$. Hence (i) holds. Q.E.D.

To show that condition (i) is rather strong, we give

EXAMPLE 1.6. Let $M = \mathbb{B}(\mathscr{H})$ be a factor of type I_∞ and T a densely defined strictly positive (i.e., $\langle T\xi, \xi\rangle \geqslant \lambda\|\xi\|^2$, $\xi \in D(T)$, for some $\lambda > 0$) symmetric operator on $\mathscr{H}$. Let Π be the directed set of all finite-dimensional projections p with $p\mathscr{H} \subset D(T)$. Letting $M_p = pMp$ for $p \in \Pi$, we have an increasing net $\{M_p\}$ of subalgebras of M with $M_p \nearrow M$. Usually T has many selfadjoint extensions with the same lower bound. In this case there are many strictly positive selfadjoint operator A on $\mathscr{H}$ satisfying

$$
\begin{aligned}
D(A^{1/2}) &\supset D(T),\\
\|A^{1/2}\xi\|^2 &= \langle T\xi, \xi\rangle, \qquad \xi \in D(T).
\end{aligned}
\tag{$*$}
$$

Moreover there is the greatest positive selfadjoint operator A_0 on $\mathscr{H}$ satisfying $(*)$ (cf. [9, Lemma 5] or [16, p. 468]). For any A as above, we get a faithful semifinite normal weight φ on M by $\varphi = \mathrm{tr}(A\,\cdot)$. Then, for each $p \in \Pi$, $\varphi \restriction M_p$ is bounded and independent of the choice of A. In this situation, it is seen as in the proof of (ii) $\Rightarrow$ (i) of Theorem 1.1 that condition (i) does not hold unless $\varphi = \varphi_0$ where $\varphi_0 = \mathrm{tr}(A_0\,\cdot)$.

The following corollary is easily verified from Theorem 1.1.

COROLLARY 1.7. *Suppose condition* (i) *in Theorem* 1.1. *Then*:

(1) *If $f \in M_*^+$ and $f \restriction M_\alpha$ is $\sigma_t^{\varphi_\alpha}$-invariant for each α, then f is σ_t^φ-invariant.*

(2) *If $f \in M_*^+$ and $f \restriction M_\alpha$ satisfies the KMS condition with respect to $\sigma_t^{\varphi_\alpha}$ for each α, then f satisfies the same with respect to σ_t^φ.*

(3) *If N is a von Neumann subalgebra of M with $N \subset \bigcap_\alpha M_\alpha$ and N is $\sigma_t^{\varphi_\alpha}$-invariant for each α, then N is σ_t^φ-invariant.*

2. DECREASING CASE

In this section we consider the convergence of modular automorphism groups for the decreasing case. Let $\{M_\alpha\}$ be a decreasing net of von Neumann subalgebras of M with $M_\infty = \bigcap_\alpha M_\alpha$ (we write $M_\alpha \searrow M_\infty$). In this case we further assume that each M_α is unital and $\varphi_\infty = \varphi \restriction M_\infty$ is semifinite, so that each $\varphi_\alpha = \varphi \restriction M_\alpha$ is semifinite. Besides the notations in Section 1, we use $\mathfrak{n}_\infty$, $\mathfrak{m}_\infty$, $(\mathscr{H}_\infty, \pi_\infty)$, Δ_∞, and P_∞ associated with φ_∞.

THEOREM 2.1. *Under the above assumptions, if the condition*

(i) $P_\alpha \searrow P_\infty$ *(i.e.,* $\bigcap_\alpha \mathscr{H}_\alpha = \mathscr{H}_\infty$*)*

is satisfied, then $\sigma_t^{\varphi_\alpha}(x)$ *converges strongly to* $\sigma_t^{\varphi_\infty}(x)$ *for every* $x \in M_\infty$ *where the convergence is uniform for* t *in any finite interval.*

Proof. For each $f \in M_*^+$ with $f \leqslant \varphi$, besides $h_\alpha \in \mathfrak{m}_\alpha$ we take a unique $h_\infty \in \mathfrak{m}_\infty$ with $0 \leqslant h_\infty \leqslant 1$ such that

$$f(x) = \tfrac{1}{2}\varphi(h_\infty x + x h_\infty), \qquad x \in \mathfrak{n}_\infty \cap \mathfrak{n}_\infty^*.$$

By Lemma 1.2 we get

$$\Delta_\alpha (h_\alpha)_\varphi = 2P_\alpha T\eta - (h_\alpha)_\varphi,$$
$$\Delta_\infty (h_\infty)_\varphi = 2P_\infty T\eta - (h_\infty)_\varphi.$$

As in the proof of Lemma 1.3, it can be proved that $\|(h_\alpha)_\varphi - (h_\infty)_\varphi\| \to 0$. By virtue of condition (i), we get also $\|\Delta_\alpha (h_\alpha)_\varphi - \Delta_\infty (h_\infty)_\varphi\| \to 0$. Now let $\tilde{\Delta}_\alpha = \Delta_\alpha P_\alpha + (1 - P_\alpha)$ and $\tilde{\Delta}_\infty = \Delta_\infty P_\infty + (1 - P_\infty)$. We then have

$$\|\{(1 + \tilde{\Delta}_\alpha)^{-1} - (1 + \tilde{\Delta}_\infty)^{-1}\}(1 + \Delta_\infty)(h_\infty)_\varphi\| \to 0$$

as in the proof of (i) $\Rightarrow$ (ii) of Theorem 1.1. Since

$$(1 + \Delta_\infty)(h_\infty)_\varphi = 2P_\infty T\eta = P_\infty (1 + \Delta_\varphi)\, h_\varphi,$$

it follows from Lemma 1.4 that the set $\{(1 + \Delta_\infty)(h_\infty)_\varphi\}$ is total in $\mathscr{H}_\infty$. Hence $(1 + \tilde{\Delta}_\alpha)^{-1} P_\infty \to (1 + \tilde{\Delta}_\infty)^{-1} P_\infty$ strongly, so that

$$(1 + \tilde{\Delta}_\alpha)^{-1} = (1 + \tilde{\Delta}_\alpha)^{-1} P_\infty + (1 + \tilde{\Delta}_\alpha)^{-1}(P_\alpha - P_\infty) + \tfrac{1}{2}(1 - P_\alpha)$$

converges strongly to

$$(1 + \tilde{\Delta}_\infty)^{-1} = (1 + \tilde{\Delta}_\infty)^{-1} P_\infty + \tfrac{1}{2}(1 - P_\infty).$$

Therefore, for every $x \in M_\infty$ and $\xi \in \mathscr{H}_\infty$, we have

$$\begin{aligned} &\|\pi_\varphi(\sigma_t^{\varphi_\alpha}(x))\,\xi - \pi_\varphi(\sigma_t^{\varphi_\infty}(x))\,\xi\| \\ &\quad = \|\tilde{\Delta}_\alpha^{it} \pi_\varphi(x)\, \tilde{\Delta}_\alpha^{-it} \xi - \tilde{\Delta}_\infty^{it} \pi_\varphi(x)\, \tilde{\Delta}_\infty^{-it} \xi\| \to 0 \end{aligned}$$

uniformly for t in any finite interval. Choosing a net $\{a_j\}$ in $\mathfrak{m}_\infty$ with $0 \leqslant a_j \nearrow 1$, we have

$$J_\varphi \pi_\varphi(x) J_\varphi (a_j)_\varphi = \pi_\varphi(a_j)\, J_\varphi x_\varphi \to J_\varphi x_\varphi, \qquad x \in \mathfrak{n}_\varphi.$$

This yields that $\pi_\varphi(M)'\mathscr{H}_\infty$ is dense in $\mathscr{H}_\varphi$. Hence we get the desired conclusion. Q.E.D.

REMARKS 2.2. (1) Suppose condition (i) in Theorem 1.1 (resp. Theorem 2.1). If $\{x_\alpha\}$ is a bounded net in M such that $x_\alpha \in M_\alpha$ and $x_\alpha \to x$ strongly, then $\sigma_t^{\varphi_\alpha}(x_\alpha)$ converges strongly to $\sigma_t^\varphi(x)$ (resp. $\sigma_t^{\varphi_\infty}(x)$) uniformly for t in any finite interval.

(2) If condition (i) in Theorem 1.1 (resp. Theorem 2.1) is satisfied, then the same holds for any weight ψ on M with $\lambda^{-1}\varphi \leqslant \psi \leqslant \lambda\varphi$ for some $\lambda > 0$.

(3) The converse in Theorem 2.1 does not hold, even if φ is bounded. Indeed we can find a von Neumann algebra M with a cyclic and separating vector ξ and a decreasing net $\{M_\alpha\}$ of von Neumann subalgebras of M such that ξ is cyclic for each M_α and $M_\infty = \bigcap_\alpha M_\alpha = \mathbb{C}1$ (cf. [5, Example 1]). Let $\varphi(x) = \langle x\xi, \xi\rangle$ for $x \in M$, then $\mathscr{H}_\alpha = \mathscr{H}_\varphi$ for each α and $\mathscr{H}_\infty = \mathbb{C}\xi$. Thus condition (i) in Theorem 2.1 is not satisfied, but the conclusion trivially holds.

3. CONVERGENCE OF CONNES COCYCLE DERIVATIVES

Let φ and ψ be faithful semifinite normal weights on M. We define the faithful semifinite normal weight ω on $M \otimes F_2$ by

$$\omega\left(\sum x_{ij} \otimes e_{ij}\right) = \varphi(x_{11}) + \psi(x_{22}), \qquad \sum x_{ij} \otimes e_{ij} \in (M \otimes F_2)_+,$$

where F_2 is the factor of type I_2 and $(e_{ij})_{i,j=1,2}$ is the natural basis for F_2. Then

$$\begin{aligned} \mathfrak{n}_\omega &= \mathfrak{n}_\varphi \otimes e_{11} + \mathfrak{n}_\psi \otimes e_{12} + \mathfrak{n}_\varphi \otimes e_{21} + \mathfrak{n}_\psi \otimes e_{22}, \\ \mathfrak{n}_\omega \cap \mathfrak{n}_\omega^* &= (\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*) \otimes e_{11} + (\mathfrak{n}_\psi \cap \mathfrak{n}_\varphi^*) \otimes e_{12} \\ &\quad + (\mathfrak{n}_\varphi \cap \mathfrak{n}_\psi^*) \otimes e_{21} + (\mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*) \otimes e_{22}. \end{aligned}$$

In the GNS representation of $M \otimes F_2$ induced by ω, the Hilbert space $\mathscr{H}_\omega$ is given by

$$\mathscr{H}_\omega = \mathscr{H}_\varphi \oplus \mathscr{H}_\psi \oplus \mathscr{H}_\varphi \oplus \mathscr{H}_\psi$$

with the canonical injection

$$X_\omega = (x_{11})_\varphi \oplus (x_{12})_\psi \oplus (x_{21})_\varphi \oplus (x_{22})_\psi, \qquad X = \sum x_{ij} \otimes e_{ij} \in \mathfrak{n}_\omega.$$

The Connes cocycle derivative $(D\psi : D\varphi)_t$ is given by

$$\sigma_t^{\omega}(1 \otimes e_{21}) = (D\psi : D\varphi)_t \otimes e_{21}, \qquad t \in \mathbb{R}.$$

Moreover we define

$$\sigma_t^{\psi,\varphi}(x) = (D\psi : D\varphi)_t \sigma_t^{\varphi}(x) = \sigma_t^{\psi}(x)(D\psi : D\varphi)_t, \qquad x \in M, \quad t \in \mathbb{R}.$$

Then $\sigma_t^{\psi,\varphi}$ is a strongly continuous one-parameter group of isometries on M and we have $\sigma_t^{\omega}(x \otimes e_{21}) = \sigma_t^{\psi,\varphi}(x) \otimes e_{21}$. The relative modular operator $\Delta_{\psi,\varphi}$ on $\mathscr{H}_\varphi$ is defined through the polar decomposition $S_{\psi,\varphi} = J_{\psi,\varphi}\Delta_{\psi,\varphi}^{1/2}$ of the closure $S_{\psi,\varphi}$ of the closable conjugate linear operator $x_\varphi \mapsto (x^*)_\psi$, $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\psi^*$. It is easily checked that

$$\Delta_\omega = \Delta_\varphi \oplus \Delta_{\varphi,\psi} \oplus \Delta_{\psi,\varphi} \oplus \Delta_\psi .$$

Now let $M_\alpha \nearrow M$ and assume that $\varphi_\alpha = \varphi \restriction M_\alpha$ and $\psi_\alpha = \psi \restriction M_\alpha$ are semifinite for each α. Then $M_\alpha \otimes F_2 \nearrow M \otimes F_2$ and $\omega_\alpha = \omega \restriction (M_\alpha \otimes F_2)$ is semifinite for each α. It is readily seen that condition (i) in Theorem 1.1 for ω is equivalent to the following:

(a) $\bigcup_\alpha \{x_\varphi : x \in \mathfrak{n}_{\varphi_\alpha} \cap \mathfrak{n}_{\varphi_\alpha}^*\}$ is a core of $\Delta_\varphi^{1/2}$,
(b) $\bigcup_\alpha \{x_\psi : x \in \mathfrak{n}_{\psi_\alpha} \cap \mathfrak{n}_{\varphi_\alpha}^*\}$ is a core of $\Delta_{\varphi,\psi}^{1/2}$,
(c) $\bigcup_\alpha \{x_\varphi : x \in \mathfrak{n}_{\varphi_\alpha} \cap \mathfrak{n}_{\psi_\alpha}^*\}$ is a core of $\Delta_{\psi,\varphi}^{1/2}$,
(d) $\bigcup_\alpha \{x_\psi : x \in \mathfrak{n}_{\psi_\alpha} \cap \mathfrak{n}_{\psi_\alpha}^*\}$ is a core of $\Delta_\psi^{1/2}$.

THEOREM 3.1. *Let $M_\alpha \nearrow M$ and φ, ψ be as above. If conditions* (a)–(d) *hold (particularly if φ and ψ are bounded), then $\sigma_t^{\psi_\alpha,\varphi_\alpha}(x)$ converges strongly to $\sigma_t^{\psi,\varphi}(x)$ for every $x \in \bigcup_\alpha M_\alpha$ and $(D\psi_\alpha : D\varphi_\alpha)_t$ converges strongly to $(D\psi : D\varphi)_t$, where the convergence is uniform for t in any finite interval.*

Proof. The first part is immediate from Theorem 1.1 applied to $M_\alpha \otimes F_2 \nearrow M \otimes F_2$ and ω. For the second part, let e_α be the unity of M_α, then $e_\alpha \nearrow 1$. By Remark 2.2(1), it follows that $\sigma_t^{\omega_\alpha}(e_\alpha \otimes e_{21}) = (D\psi_\alpha : D\varphi_\alpha)_t \otimes e_{21}$ converges strongly to $\sigma_t^{\omega}(1 \otimes e_{21}) = (D\psi : D\varphi)_t \otimes e_{21}$ uniformly for t in any finite interval. Q.E.D.

For the decreasing case, we have

THEOREM 3.2. *Let $M_\alpha \searrow M_\infty$ where each M_α is unital. Assume that $\varphi_\infty = \varphi \restriction M_\infty$ and $\psi_\infty = \psi \restriction M_\infty$ are semifinite. If $\bigcap_\alpha \mathscr{H}_{\varphi_\alpha} = \mathscr{H}_{\varphi_\infty}$ and $\bigcap_\alpha \mathscr{H}_{\psi_\alpha} = \mathscr{H}_{\psi_\infty}$, then $\sigma_t^{\psi_\alpha,\varphi_\alpha}(x)$ converges strongly to $\sigma_t^{\psi_\infty,\varphi_\infty}(x)$ for every $x \in M_\infty$ and in particular $(D\psi_\alpha : D\varphi_\alpha)_t$ converges strongly to $(D\psi_\infty : D\varphi_\infty)_t$. The convergence is uniform for t in any finite interval.*

Proof. Obviously $M_\alpha \otimes F_2 \searrow M_\infty \otimes F_2$ and $\omega_\infty = \omega \restriction (M_\infty \otimes F_2)$ is semifinite. Since the condition $\bigcap_\alpha \mathscr{H}_{\omega_\alpha} = \mathscr{H}_{\omega_\infty}$ is equivalent to $\bigcap_\alpha \mathscr{H}_{\varphi_\alpha} = \mathscr{H}_{\varphi_\infty}$ and $\bigcap_\alpha \mathscr{H}_{\psi_\alpha} = \mathscr{H}_{\psi_\infty}$, the theorem follows immediately from Theorem 2.1. Q.E.D.

4. Connections with Martingale Convergence

The martingale convergence theorems in von Neumann algebras have been developed by several authors [8, 10, 13, 19, 20]. In this section we discuss the convergence of Connes cocycle derivatives in connection with the martingale convergence.

Let φ be a faithful semifinite normal weight on M and N a unital von Neumann subalgebra of M. The conditional expectation $\varepsilon: M \to N$ with respect to φ is a unique norm one normal projection ε of M onto N such that $\varphi(x) = \varphi(\varepsilon(x))$ for all $x \in M_+$. According to Takesaki [18], there exists the conditional expectation $\varepsilon: M \to N$ with respect to φ if and only if $\varphi \restriction N$ is semifinite and $\sigma_t^\varphi(N) = N$ for every $t \in \mathbb{R}$. In this case, ε is determined by $\pi_\psi(\varepsilon(x)) = P\pi_\varphi(x) P$ for all $x \in M$ where $\psi = \varphi \restriction N$ and P is the orthogonal projection of $\mathscr{H}_\varphi$ onto $\mathscr{H}_\psi$. If $x \in \mathfrak{n}_\varphi$, then we have $\varepsilon(x) \in \mathfrak{n}_\psi$ and $(\varepsilon(x))_\psi = Px_\varphi$.

Let $M_\alpha \nearrow M$ (or $M_\alpha \searrow M_\infty$) where each M_α is unital. In the following theorems we assume that for each α there exists the conditional expectation $\varepsilon_\alpha : M \to M_\alpha$ with respect to φ. For the case $M_\alpha \searrow M_\infty$, we further assume that $\varphi_\infty = \varphi \restriction M_\infty$ is semifinite, so that the conditional expectation $\varepsilon_\infty : M \to M_\infty$ with respect to φ exists. Then $\{\varepsilon_\alpha\}$ becomes an increasing (or decreasing) martingale of conditional expectations. Note that the convergence properties in Theorems 1.1 and 2.1 hold under these assumptions since $\sigma_t^{\varphi_\alpha} = \sigma_t^\varphi \restriction M_\alpha$ for each α (cf. [7, Lemma 1.4.3]). We here mention the martingale convergence theorem given by Tsukada [19] as follows.

THEOREM 4.1. (1) *Let $M_\alpha \nearrow M$. Then $\varepsilon_\alpha(x)$ converges strongly to x for every $x \in M$ and $f \circ \varepsilon_\alpha$ converges in the norm to f for every $f \in M_*$.*

(2) *Let $M_\alpha \searrow M_\infty$. Then $\varepsilon_\alpha(x)$ converges strongly to $\varepsilon_\infty(x)$ for every $x \in M$ and $f \circ \varepsilon_\alpha$ converges in the norm to $f \circ \varepsilon_\infty$ for every $f \in M_*$.*

THEOREM 4.2. *Let $M_\alpha \nearrow M$ and ψ be a faithful semifinite normal weight on M. Then $(D\psi_\alpha : D\varphi_\alpha)_t$ converges strongly to $(D\psi : D\varphi)_t$ uniformly for t in any finite interval in the following cases:*

(i) *$\lambda^{-1}\varphi \leqslant \psi \leqslant \lambda\varphi$ for some $\lambda > 0$,*

(ii) *ψ is bounded.*

Proof. Since $\sigma_t^{\varphi_\alpha} = \sigma_t^\varphi \upharpoonright M_\alpha$ for each α, the conditions in Theorem 1.1 hold for φ. For case (i), it is obvious that conditions (a)–(d) in Section 3 are satisfied. Hence case (i) follows from Theorem 3.1. For case (ii), Theorem 4.1(1) yields that $\|\psi \circ \varepsilon_\alpha - \psi\| \to 0$. By [7, Lemma 1.4.4], we have

$$(D\psi_\alpha : D\varphi_\alpha)_t = (D(\psi \circ \varepsilon_\alpha) : D(\varphi \circ \varepsilon_\alpha))_t = (D(\psi \circ \varepsilon_\alpha) : D\varphi)_t.$$

Hence we get the conclusion by [16, Proposition 7.18]. Q.E.D.

THEOREM 4.3. *Let $M_\alpha \searrow M_\infty$ and ψ be a faithful semifinite normal weight on M. Then $(D\psi_\alpha : D\varphi_\alpha)_t$ converges strongly to $(D\psi_\infty : D\varphi_\infty)_t$ uniformly for t in any finite interval in the following cases:*

(i) *$\lambda^{-1}\varphi \leqslant \psi \leqslant \lambda\varphi$ for some $\lambda > 0$,*

(ii) *the conditional expectation of M onto M_α with respect to ψ exists for each α and ψ_∞ is semifinite,*

(iii) *ψ is bounded.*

Proof. Let $x \in \mathfrak{n}_\varphi$. Since $(\varepsilon_\alpha(x))_\varphi = P_\alpha x_\varphi$, $(\varepsilon_\infty(x))_\varphi = P_\infty x_\varphi$, and $\varepsilon_\alpha(x) \to \varepsilon_\infty(x)$ strongly by Theorem 4.1(2), we have $\|P_\alpha x_\varphi - P_\infty x_\varphi\| \to 0$. Hence $\bigcap_\alpha \mathscr{H}_{\varphi_\alpha} = \mathscr{H}_{\varphi_\infty}$. For cases (i) and (ii), also $\bigcap_\alpha \mathscr{H}_{\psi_\alpha} = \mathscr{H}_{\psi_\infty}$ holds. Hence Theorem 3.2 gives the conclusion. Case (iii) is shown as in case (ii) of Theorem 4.2 from Theorem 4.1(2) and [16, Proposition 7.18]. Q.E.D.

EXAMPLE 4.4. *Let* $(\Omega, \mathscr{F})$ be a measurable space and μ, ν be equivalent finite measures on $\mathscr{F}$. Let $\{\mathscr{F}_\alpha\}$ be an increasing (resp. decreasing) net of sub-σ-fields of $\mathscr{F}$ with $\mathscr{F} = \bigvee_\alpha \mathscr{F}_\alpha$ (resp. $\mathscr{F}_\infty = \bigcap_\alpha \mathscr{F}_\alpha$). We take $M = L^\infty(\Omega, \mathscr{F}, \mu)$ and $M_\alpha = L^\infty(\Omega, \mathscr{F}_\alpha, \mu)$. The measures μ, ν naturally define faithful normal positive linear functionals φ, ψ on M. Taking the Radon–Nikodym derivative $f = d\nu/d\mu$, we have $(D\psi : D\varphi)_t = f^{it}$ and $(D\psi_\alpha : D\varphi_\alpha)_t = E_\mu(f \mid \mathscr{F}_\alpha)^{it}$ where $E_\mu(f \mid \mathscr{F}_\alpha)$ is the conditional expectation of f with respect to $\mathscr{F}_\alpha$ and μ. Hence Theorem 4.2 (resp. Theorem 4.3) asserts that

$$\|E_\mu(f \mid \mathscr{F}_\alpha)^{it} - f^{it}\|_{L^2} \to 0$$
$$(\text{resp. } \|E_\mu(f \mid \mathscr{F}_\alpha)^{it} - E_\mu(f \mid \mathscr{F}_\infty)^{it}\|_{L^2} \to 0)$$

uniformly for t in any finite interval. This is a version of the classical martingale convergence theorem.

Recently Accardi and Cecchini [1] generalized the concept of conditional expectations on von Neumann algebras. The Accardi and Cecchini generalized conditional expectation $\varepsilon: M \to N$ with respect to φ always exists whenever N is a unital von Neumann subalgebra and $\varphi \upharpoonright N$ is semifinite, but

ε is not necessarily a projection onto N and lacks a useful property that $\varepsilon(axb) = a\varepsilon(x)\, b$ for $a, b \in N$ and $x \in M$.

In [12] we have studied the strong martingale convergence of generalized conditional expectations. For the case $M_\alpha \nearrow M$, the conditions in Theorem 1.1 hold if and only if $\varepsilon_\alpha(x) \to x$ strongly for every $x \in M$ where $\varepsilon_\alpha : M \to M_\alpha$ is the generalized conditional expectation with respect to φ. For the case $M_\alpha \searrow M_\infty$, if condition (i) in Theorem 2.1 is satisfied, then $\varepsilon_\alpha(x) \to \varepsilon_\infty(x)$ strongly for every $x \in M$. See [12] for detailed arguments.

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